

Phase controllable dynamical localization: a generalization of the Dunlap-Kenkre result

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Dunlap-Kenkre result states that Dynamical Localization (DL) of a field driven quantum particle in a discrete periodic lattice happens when the ratio of the field magnitude to the field frequency (say, η) of the diagonal sinusoidal drive is a root of the ordinary Bessel function of order 0. This has been experimentally verified. A generalization of the Dunlap-Kenkre result is presented here. We analytically show that if we have an off-diagonal driving field (with modulation δ) and diagonal driving field with different frequencies (say ω_1 and ω_2 respectively) and a definite phase relationship ϕ between them, one can obtain DL if (1) η is a zero of the Bessel function of order 0 and ϕ is an odd multiple of $\pi/2$ for equal and $\frac{\omega_1}{\omega_2} = \text{odd integer}$ driving frequencies, (2) η is a zero of the Bessel function of order 0 and ϕ is an integer multiple of π including zero for $\frac{\omega_1}{\omega_2} = \text{even integer} \equiv m$, and (3) $\phi = -\arcsin(\frac{J_0(\eta)}{\delta J_m(\eta)})$ and η is not a zero of the Bessel function of the even order m .

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Needless to repeat the popular idea that the cold atoms in optical lattices provide a well controllable experimental apparatus for testing the models of condensed matter physics[1]. Recently, in addition to various other experimental verifications, the phenomenon of Dynamical Localization (DL) has been realized with cold atoms in optical lattices[2]. DL has been previously predicted in seminal work of Dunlap and Kenkre[3]. DL states that the wave packet of a single particle moving in a single-band tight-binding lattice with nearest-neighbor coupling driven by a spatially homogeneous ac field becomes localized whenever the ratio of the field magnitude to the field frequency is a root of the ordinary Bessel function of order 0. This effect is latter understood physically in terms of dynamical band collapse[4] with far reaching consequences including metal-insulator transitions in quasi-periodic lattices[5]. Dunlap-Kenkre result has been recently generalized for arbitrary time-periodic forcing and going beyond nearest-neighbor approximation[6]. DL has also been predicted in semiconductor superlattices[7]. Recently the importance of phase of driving field has been realized[8] which show that, for systems with strong attractive pairing, it enables different types of collisions and re-collisions between paired and un-paired atoms.

In this letter we point out the effect of phase difference between previously introduced off-diagonal drive[9] and diagonal drive[3] on the phenomenon of dynamical localization. We analytically see that DL can be controlled with various other experimental control parameters and new DL conditions exist. Our results are readily amenable to experiments.

The Hamiltonian of an atom in an amplitude modulated and driven optical lattice (see Fig. 1) in one dimension is given as,

$$\hat{H}(t) = -\frac{V}{2}(1 + \delta \sin(\omega_1 t + \phi)) \sum_l (|l\rangle\langle l+1| + |l+1\rangle\langle l|)$$

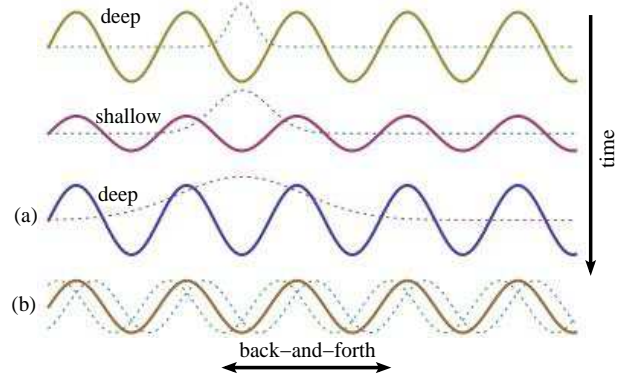


FIG. 1: Modulation of the optical lattice. The upper figure (a) corresponds to the “deep-shallow-deep” periodic modulation of the optical lattice (off-diagonal term in the Hamiltonian with frequency ω_1). The lower figure (b) represents the back-and-forth motion (shaking) of the optical lattice with frequency ω_2 (this corresponds to the diagonal term in the Hamiltonian). The dotted line in the upper figure shows a schematic time evolution of the atomic wave packet.

$$+\xi \cos(\omega_2 t) \sum_l |l\rangle\langle l|. \quad (1)$$

Where ξ and ω_2 is the strength and frequency of the diagonal drive (equivalent to shaking the optical lattice back and forth). δ and ω_1 is the strength and frequency of the off-diagonal modulation (by periodically modulating the amplitude of the optical wells—“deep-shallow-deep” periodic modulation). ϕ is the phase difference between the drives and V is the tunneling matrix element between the nearest neighbor optical wells. $|l\rangle$ is the Wannier state localized on lattice site l (lattice constant = 1). For $\delta = 0$ we have the case considered by Dunlap and Kenkre.

We start by putting the atom at a lattice site 0 (Fig. 1). Thus density matrix of the atom at $t = 0$ in site representation is $\rho_{m,n}(t = 0) = \delta_{m,0}\delta_{n,0}$. The time evolution of the atomic wave packet is given by Liouville-Von Neu-

TABLE I: Summary of results

$\omega_1 = \omega_2$	$\frac{\omega_1}{\omega_2} = m$ (odd integer)	$\frac{\omega_1}{\omega_2} = m$ (even integer)
$t^2 \frac{\beta^2}{2} (J_0^2(\eta) + \delta^2 \cos^2 \phi J_1^2(\eta))$	$t^2 \frac{\beta^2}{2} (J_0^2(\eta) + \delta^2 \cos^2 \phi J_m^2(\eta))$	$t^2 \frac{\beta^2}{2} (J_0(\eta) + \delta \sin \phi J_m(\eta))^2$
η is a zero of J_0 and $\phi = (2n+1)\frac{\pi}{2}$ (DL condition)	η is a zero of J_0 and $\phi = (2n+1)\frac{\pi}{2}$ (DL condition)	η is a zero of J_0 and $\phi = n\pi$ or $\phi = -\sin^{-1}(\frac{J_0(\eta)}{\delta J_m(\eta)})$ and η is not a zero of J_m .

mann equation

$$i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = [\hat{H}, \hat{\rho}]. \quad (2)$$

In the site representation it reads,

$$\begin{aligned} \frac{\partial \langle m | \hat{\rho}(t) | n \rangle}{\partial t} &= \frac{\partial \rho_{m,n}(t)}{\partial t} = \frac{i\beta}{2} (1 + \delta \sin(\omega_1 t + \phi)) \\ &\times [\rho_{m+1,n} - \rho_{m,n+1} + \rho_{m-1,n} - \rho_{m,n-1}] \\ &- i\lambda \cos(\omega_2 t) (m - n) \rho_{m,n}(t). \end{aligned} \quad (3)$$

Here $\beta \equiv V/\hbar$, and $\lambda \equiv \xi/\hbar$. Consider the following transformation

$$\rho_{m,n} = \bar{\rho}_{m,n} e^{-if(t)(m-n)}, \quad f(t) = \frac{\lambda}{\omega_2} \sin(\omega_2 t). \quad (4)$$

$$\begin{aligned} \frac{\partial \bar{\rho}_{m,n}}{\partial t} &= \frac{i\beta}{2} (1 + \delta \sin(\omega_1 t + \phi)) \\ &\times (e^{-if(t)} \bar{\rho}_{m+1,n} - e^{if(t)} \bar{\rho}_{m,n+1} + e^{if(t)} \bar{\rho}_{m-1,n} \\ &- e^{-if(t)} \bar{\rho}_{m,n-1}). \end{aligned} \quad (5)$$

On writing the above equation in Fourier space with,

$$\tilde{\rho}(k_1, k_2, t) = \sum_{m,n=-\infty}^{+\infty} \bar{\rho}(t) e^{-imk_1} e^{ink_2}, \quad (6)$$

puts it into a much simpler form,

$$\begin{aligned} \frac{\partial \tilde{\rho}(k_1, k_2, t)}{\partial t} &= 2i\beta (1 + \delta \sin(\omega_1 t + \phi)) \\ &\times \sin\left(\frac{k_1 + k_2}{2} - f(t)\right) \sin\left(\frac{k_2 - k_1}{2}\right) \tilde{\rho}(k_1, k_2, t). \end{aligned} \quad (7)$$

This can be further simplified by defining center-of-mass and relative coordinates as $p \equiv \frac{k_1 + k_2}{2}$, and $u \equiv k_1 - k_2$, and re-defining $\tilde{\rho}(k_1, k_2, t) \equiv \varrho(p, u, t)$, whose solution is straightforward

$$\varrho(p, u, t) = e^{-2i\beta \sin(u/2) \int_0^t (1 + \delta \sin(\omega_1 t' + \phi)) \sin(p - f(t')) dt'}. \quad (8)$$

Where we have used $\varrho(p, u, 0) = \sum_{m,n} \delta_{m,0} \delta_{n,0} e^{+i(\lambda/\omega_2) \sin(\omega_2 t)(m-n) - imk_1 + ink_2} = 1$. Now on integrating the above equation with respect to p we obtain a closed equation in $\xi(u, t) \equiv \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varrho(p, u, t) dp$.

We are interested in finding the mean displacement and mean-squared displacement of the atom from the

starting point. One notices that $\langle n(t) \rangle = i \frac{\partial \xi(u, t)}{\partial u} \Big|_{u=0} = i \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \sum_{m,n} \bar{\rho}_{m,n}(t) \frac{\partial}{\partial u} e^{-im(p+u/2) + in(p-u/2)} \Big|_{u=0} = \sum_n n \rho_{n,n}(t)$. Similarly $\langle n^2(t) \rangle = -\frac{\partial^2 \xi(u, t)}{\partial u^2} \Big|_{u=0}$.

A simple computation using the above prescription show that mean displacement is always zero (centre-of-mass of the wave packet does not move). A computation of the mean-squared displacement leads to

$$\langle n^2(t) \rangle = \frac{\beta^2}{2} (u^2(t) + v^2(t)), \quad (9)$$

with $u(t) = \int_0^t dt' \cos(\eta \sin(\omega_2 t')) + \delta \int_0^t dt' \cos(\eta \sin(\omega_2 t')) \sin(\omega_1 t' + \phi)$, $\eta = \frac{\lambda}{\omega_2}$ and $v(t) = \int_0^t dt' \sin(\eta \sin(\omega_2 t')) + \delta \int_0^t dt' \sin(\eta \sin(\omega_2 t')) \sin(\omega_1 t' + \phi)$.

For the case of equal frequencies $\omega_1 = \omega_2$ we write $u(t)$ and $v(t)$ in terms of time bounded ($B_u(t)$ and $B_v(t)$) and time unbounded functions as $B_u(t) + tJ_0(\eta)$ and $B_v(t) + t\delta \cos(\phi)J_1(\eta)$. We obtain (after a long calculation) the mean-squared displacement in the long time limit $t \gg \frac{2\pi}{\omega}$ as

$$\langle n^2(t) \rangle = t^2 \frac{\beta^2}{2} (J_0(\eta)^2 + \delta^2 \cos^2 \phi J_1(\eta)^2). \quad (10)$$

Here J_0 and J_1 are the ordinary Bessel functions of order 0 and 1 and $\eta \geq 0$. For $\delta = 0$ we get back the Dunlap-Kenkre result, as we should. Here we get DL if η is the zero of J_0 and $\phi = (2n+1)\frac{\pi}{2}$, $n \in \mathcal{Z}$. One important implication is that the motion of the atom is now controllable through δ and phase ϕ also.

For the case of unequal frequencies $\frac{\omega_1}{\omega_2} = m$ (integer) one sees that the analysis can be further divided into two sub-cases (1) m odd integer, and (2) m even integer. As before, writing $u(t)$ and $v(t)$ in terms of time bounded and time unbounded functions, we obtain $\langle n^2(t) \rangle$ for the first sub-case (m odd integer) in the long time limit $t \gg \frac{2\pi}{\omega}$ as

$$\langle n^2(t) \rangle = t^2 \frac{\beta^2}{2} (J_0(\eta)^2 + \delta^2 \cos^2 \phi J_m(\eta)^2), \quad (11)$$

and in the second sub-case (m even integer) as

$$\langle n^2(t) \rangle = t^2 \frac{\beta^2}{2} (J_0(\eta) + \delta \sin \phi J_m(\eta))^2. \quad (12)$$

Here J_m is the ordinary Bessel functions of order $m = \frac{\omega_1}{\omega_2}$ and $\eta \geq 0$. For $\delta = 0$ we again get back the Dunlap-Kenkre result as we should. Here we get DL (1) if η is

the zero of J_0 and $\phi = (2n+1)\frac{\pi}{2}$, $n \in \mathcal{Z}$ (odd integer m case), (2) if η is a zero of J_0 and ϕ is an integer multiple of π including zero (even integer m case), and (3) if $\phi_{critical} = -\arcsin(\frac{J_0(\eta)}{\delta J_m(\eta)})$ and η is not a zero of the Bessel function of the *even* order m . These results are summarized in table I.

There is another remarkable dynamical result. If diagonal drive is zero i.e., $\xi = 0$ i.e., $\eta = 0$, the off-diagonal drive $\delta \sin(\omega_1 t + \phi)$ has *no effect* on the temporal evolution of the atomic wave packet. The width of the wave packet expand $\propto t^2$ as it should in pure quantum dynamics (it will not see the “deep-shallow-deep” motion of the optical lattice!).

The above mathematical results should be directly verifiable with present sophistication of experiments with cold atoms in optical lattices[2] especially the $\phi_{critical} = -\sin^{-1}(\frac{J_0(\eta)}{\delta J_m(\eta)})$.

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